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The Proof of Two Identities by Using the Method of Generating Function

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Abstract: In this paper, the method of generating function combined with power series is applied to verify the correctness of two identities which are widely used in engineering technology.
Keywords: Generating function method; Identity; Power series; Coefficients

INTRODUCTION

In engineering, the following two identities are often used

$$\sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k=\frac{N}{16} \qquad (1)$$

$$\sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k^2=\frac{N}{16}(\frac{N}{6}+1) \qquad (2)$$

SOLVING PROCESS

As we know , $(1+x)^\alpha$ can be represented as Taylor series as follows

$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots,\alpha\in(-1,1) \qquad (3)$$

Let $\alpha=-\frac{1}{2}$,we obtained

$$(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{1\cdot3}{2\cdot4}x^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3+\cdots \qquad (4)$$

and

$$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1\cdot3}{2\cdot4}x^2+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^3+\cdots=\sum_{n=0}^{\infty}C_{2n}^n(\frac{x}{4})^n \qquad (5)$$

$$(1-4x)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}C_{2n}^n x^n \qquad (6)$$

Take a derivative of both sides of Eq.(6) with respect to x and get

$$2(1-4x)^{-3/2}=\sum_{n=1}^{\infty}nC_{2n}^n x^{n-1} \qquad (7)$$

Multiply Eq.(6) with Eq.(7) .and gain

$$2x(1-4x)^{-2}=(\sum_{n=0}^{\infty}C_{2n}^n x^n)(\sum_{n=1}^{\infty}nC_{2n}^n x^{n-1})=\sum_{n=0}^{\infty}(\sum_{k=0}^{n}kC_{2k}^k C_{2(n-k)}^{n-k})x^n \qquad (8)$$

Because of

$$\frac{1}{1-4x}=\sum_{n=0}^{\infty}(4x)^n \qquad (9)$$

Take a derivative of both sides of Eq.(9) with respect to x and get

$$\frac{4}{(1-4x)^2}=\sum_{n=0}^{\infty}n(4x)^{n-1} \qquad (10)$$

Hence

$$\frac{2x}{(1-4x)^2}=\sum_{n=0}^{\infty}n(4x)^{n-1}\frac{x}{2}=\sum_{n=0}^{\infty}n2^{2n-3}x^n \qquad (11)$$

Compare Eq.(8) with Eq.(11) and get

$$\sum_{k=0}^{n}kC_{2k}^{k}C_{2(n-k)}^{n-k}=n2^{2n-3} \qquad (12)$$

$$\sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k=\sum_{K=0}^{n}\frac{(2n-2k)!(2k)!}{2^{2n}[k!(n-k)!]^2}k=\frac{\sum_{K=0}^{n}kC_{2k}^{k}C_{2(n-k)}^{n-k}}{2^{2n}}=\frac{n2^{2n-3}}{2^{2n}}=\frac{n}{8}=\frac{N}{16} \qquad (13)$$

Take a derivative of both sides of Eq.(8) with respect to x and get

$$2(1-4x)^{-3/2}+2x(-\frac{3}{2})(1-4x)^{-5/2}(-4)=\sum_{n=1}^{\infty}n^2C_{2n}^{n}x^{n-1} \qquad (14)$$

Multiply Eq.(14) with Eq.(6) .and gain

$$2(1-4x)^{-2}+12x(1-4x)^{-3}=(\sum_{n=1}^{\infty}n^2C_{2n}^{n}x^{n-1})(\sum_{n=0}^{\infty}C_{2n}^{n}x^n) \qquad (15)$$

Because of

$$\frac{1}{(1-4x)^2}=\frac{1}{4}\sum_{n=1}^{\infty}4^n nx^{n-1} \qquad (16)$$

Take a derivative of both sides of Eq.(16) with respect to x and get

$$\frac{8}{(1-4x)^3}=\frac{1}{4}\sum_{n=2}^{\infty}4^n n(n-1)x^{n-2} \qquad (17)$$

Hence

$$12x(1-4x)^{-3}=\frac{3}{2}x\frac{1}{4}\sum_{n=2}^{\infty}4^n n(n-1)x^{n-2}=\frac{3}{8}\sum_{n=2}^{\infty}4^n n(n-1)x^{n-1} \qquad (18)$$

Add Eq.(17) to Eq.(19) and find out the item x^{n-1} ,we can obtain

$$\frac{1}{2}4^n nx^{n-1}+\frac{3}{8}4^n n(n-1)x^{n-1}=\sum_{k+l=n}k^2C_{2k}^{k}x^{k-1}C_{2l}^{l}x^l=\sum_{k=0}^{n}k^2C_{2k}^{k}C_{2(n-k)}^{n-k}x^{n-1} \qquad (19)$$

So we have

$$\sum_{k=0}^{n}k^2C_{2k}^{k}C_{2(n-k)}^{n-k}=\frac{1}{2}4^n n+\frac{3}{8}4^n n(n-1)=\frac{1}{8}4^n n(3n+1) \qquad (20)$$

$$\sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k^2=\sum_{K=0}^{N/2}\frac{C_{2k}^{k}C_{N-2k}^{N/2-k}k^2}{2^N}=\frac{1}{2^N}4^{N/2}\frac{N}{2}\frac{1}{8}(3\frac{N}{2}+1)=\frac{N}{16}(\frac{N}{6}+1) \qquad (21)$$

CONCLUSIONS

Using the method of generating function, we can obtain two identities as follows

$$\sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k=\frac{N}{16} \quad \sum_{K=0}^{N/2}\frac{(N-2k)!(2k)!}{2^N[k!(N/2-k)!]^2}k^2=\sum_{K=0}^{N/2}\frac{C_{2k}^{k}C_{N-2k}^{N/2-k}k^2}{2^N}=\frac{N}{16}(\frac{N}{6}+1)$$

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REFERENCES

1. Guoyu, M.A. (1979). The Generating Function Method—A Good Way to Solve the Difference Equations. Journal of Beijing University of Chemical Technology, 4, 83-102.
2. Department of Mathematics of East China Normal University, (2010). Mathematical Analysis. Higher Education Press, 6.