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A Non-Monotone Conic Trust Region Method with Fixed Step-Size

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Abstract: In this paper, a new non-monotone conic trust region method with fixed step-size based on conic models for solving unconstrained optimization problem is proposed. Unlike the traditional trust region methods, the sub-problem of our new algorithm is the conic minimization sub-problem. Moreover, we use the fixed step-size to obtain a new point when the trial step is not accepted. Theoretical analysis indicates that the new method preserves the global convergence under suitable conditions.
Keywords: non-monotone; conic trust region method; unconstrained optimization; fixed step-size; global convergence.

INTRODUCTION

Unconstrained optimization problem as follows:

$$\min_{x \in R^n} f(x) \tag{1.1}$$

where $f(x): R^n \to R$ is a twice continuously differentiable function.

Trust region method has strong convergence and robustness, and it does not need to require the approximate Hessian matrix of the trust region sub-problem to be positive definite. So, trust region methods have been studied by many researchers [1, 2, 3].

In recent years, a variety of trust region methods have been proposed in the literatures. Nocedal and Yuan [4] presented method which combine line search and trust region method. In 2005, Mo et al. [5] proposed a fixed step length method for unconstrained optimization.

Recently, non-monotone techniques have been studied by many authors since Grippo et al. [6]. Many authors have generalized the non-monotone strategy to trust region methods and presented other new non-monotone techniques [7, 8, 9].

The traditional non-monotone trust region methods are mostly based on quadratic model, but when the objective function has strong non-quadratic, the quadratic model methods often produce a poor prediction of the minimizer of the function. In 2008, Qu et al. [10] proposed a new trust region sub-problem based on the conic model for unconstrained optimization:

$$\begin{aligned} \min \quad & c_k(s) = f_k + \frac{g_k^T s}{1-h_k^T s} + \frac{1}{2}\frac{s^T B_k s}{(1-h_k^T s)^2}, \\ s.t. \quad & 1-h_k^T s > 0, \\ & \|s\| \le \Delta_k, \end{aligned} \tag{1.2}$$

where h_k is the associated vector for conic model and it is usually called horizontal vector, Δ_k is conic trust region radius.

In our paper, we combine the sub-problem (1.2) with non-monotone technique proposed in [9] and fixed step-size to propose a new algorithm. This paper is organized as follows. In the next section, we describe our new non-monotone trust region method with fixed step-size based on conic model. The properties of this new algorithm and the global convergence property are given in Section 3. Finally, some conclusions are addressed in Section 4.

Algorithm

In this section, we describe our new non-monotone conic trust region algorithm. We obtain the trial step s_k by solving the conic model sub-problem (1.2). Then either x_{k+1} is accepted or the trust region radius is reduced according to the ratio r_k between the actual reductions of the objective function

$$Ared_k = C_k - f(x_k + s_k) \tag{2.1}$$

and the predicted reduction

$$Pred_k = -\frac{g_k^T s_k}{1-h_k^T s_k} - \frac{1}{2}\frac{s_k^T B_k s_k}{(1-h_k^T s_k)^2} \tag{2.2}$$

i.e.,
$$r_k = \frac{Ared_k(s_k)}{Pred_k(s_k)} \tag{2.3}$$

where

$$C_k = \begin{cases} f(x_k), & k=0, \\ \frac{\eta_{k-1}Q_{k-1}C_{k-1}+f(x_k)}{Q_k}, & k\ge 1, \end{cases} \quad Q_k = \begin{cases} 1, & k=0, \\ \eta_{k-1}Q_{k-1}+1, & k\ge 1, \end{cases} \tag{2.4}$$

Algorithm 2.1

Step 1. Given $x_0 \in R^n$, $\Delta_0 > 0$, $h_0 = 0$, $\mu \in (0,1)$, $\delta \in (0,1)$, $\varepsilon > 0$, $B_0 \in R^{n\times n}$ is a symmetric matrix. Set $k=0$ and choose $\eta_{\min} \in [0,1)$ and $\eta_{\max} \in [\eta_{\min},1)$.

Step 2. Compute g_k. If $\|g_k\| < \varepsilon$, stop. Otherwise, go to Step 3.

Step 3. Solve the sub-problem (1.2) for s_k. Compute C_k, $Ared_k$, $Pred_k$ and r_k.

Step 4. If $r_k \ge \mu$, set $x_{k+1} = x_k + s_k$ and go to the Step 6; otherwise, go to Step 5.

Step 5. Set $x_{k+1} = x_k + \alpha_k s_k$, where $\alpha_k = -{\delta g_k^T s_k}/{s_k^T B_k s_k}$.

Step 6. Compute Δ_{k+1} as

$$\Delta_{k+1}\begin{cases} c^p \|B_{k+1}^{-1}\|\|g_k\|, p = p+1, & if\ r_k < \mu \\ \max(c^p \|B_{k+1}^{-1}\|\|g_k\|, 4\|s_k\|, \Delta_k), & if\ r_k \ge \mu \end{cases}$$

Step 7. Update h_k [11] and the symmetric matrix B_{k+1} [11]. Set $k = k+1$, go to Step 2.

we define two index sets as below:

$$I = \{k \mid r_k \ge \mu\} \text{ and } J = \{k \mid r_k < \mu\}.$$

Convergence analysis

In this section, we will prove the global convergence property of Algorithm 2.1. The following assumptions are necessary to analyze the convergence property.

A1. The level set $L(x_0) = \{x \in R^n \mid f(x) \le f(x_0)\}$ is bounded for any given $x_0 \in R^n$.

A2. The sequences $\{B_k\}$ and $\{h_k\}$ are uniformly bounded, i.e., there exists a constant M_1 such that, $\|B_k\| \le M_1$ and $\|h_k\| \le M_1$ for all k.

A3. The $g(x)$ is Lipschitz continuous on the level set $L(x_0)$, i.e., there exists a constant τ such that, $\|g(x)-g(y)\| \le \tau \|x-y\|$.

A4. There exists a constant $\upsilon > 0$, such that $s^T B_k s \ge \upsilon s^T s$.

Remark: The constant δ of the step 1 in the Algorithm 2.1 satisfies $\delta \in (0, \min\{1, \upsilon/\tau\})$.

Lemma 1. (See Lemma 1 in [5]) Suppose that A3-A4 hold and the sequence $\{x_k\}$ is generated by Algorithm 2.1. Then for all $k \in J$, we have

$$f_{k+1} - f_k \le \frac{\delta}{2}(1-\frac{\delta\tau}{\upsilon}) g_k^T s_k \le 0 \tag{3.1}$$

Lemma 2. (See Theorem 3.1 in [10]) Suppose that A1 holds. Then there exists a positive constant $\delta_1 > 0$ such that

$$Pred_k \geq \delta_1 \|g_k\| \min\{\Delta_k, \|g_k\| / \|B_k\|\} \tag{3.2}$$

for all k, where s_k is the solution to (1.2).

Lemma 3. Suppose that A3-A4 hold and the sequence $\{x_k\}$ is generated by Algorithm 2.1. Then for all k we have

$$f_{k+1} \leq C_{k+1}. \tag{3.3}$$

And $C_{k+1} \leq C_k - \dfrac{\eta \|g_k\| \min\{\Delta_k, \|g_k\| / \|B_k\|\}}{Q_{k+1}}$, $\eta = \min\{\delta_1 \mu, \frac{\delta_1 \delta}{2}(1 - \frac{\delta \tau}{\upsilon})\}$ (3.4)

Proof. If $k \in I$, i.e., $r_k \geq \mu$. By the definition of r_k and $r_k \geq \mu$, we have

$$C_k - f_{k+1} \geq \mu Pred_k \geq \delta_1 \mu \|g_k\| \min\{\Delta_k, \frac{\|g_k\|}{\|B_k\|}\} \geq 0 \tag{3.5}$$

Thus, $C_k \geq f_{k+1}$. Then, by the definition of C_k, we obtain that

$$C_{k+1} = \frac{\eta_k Q_k C_k + f_{k+1}}{Q_{k+1}} \geq \frac{\eta_k Q_k f_{k+1} + f_{k+1}}{Q_{k+1}} = f_{k+1}. \tag{3.6}$$

So $$f_{k+1} \leq C_{k+1} \tag{3.7}$$

If $k \in J$, i.e., $r_k < \mu$. We will consider the following two cases.

Case one: $k-1 \in I$. According to Lemma 1 and (3.6), we have $C_k \geq f_k \geq f_{k+1}$. Then, using the similar process of proof, we can obtain that $C_{k+1} \geq f_{k+1}$.

Case two: $k-1 \in J$. Set $K = \{i | 1 < i \leq k, k-1 \in I\}$.

If $K \neq \varnothing$. Set $m = \min\{k : k \in K\}$, $K_1 = \{k - j | 0 \leq j \leq m-1\} \subset J$. From Lemma 1, we have $f_{k-j+1} \leq f_{k-j}$. By the definition of K, m and Formula (3.6), we obtain $f_{k-m+1} \leq C_{k-m+1}$. Suppose that $f_p \leq C_p, p \geq k-m+2$. By using induction, we can have $f_{k+1} \leq C_{k+1}$.

If $K = \varnothing$. From Lemma 1 and Formula (2.4), we obtain $f_{k+1} \leq C_{k+1}$ by using the induction.

Through the above analysis, we know Formula (3.3) is true.

Now, we prove the Formula (3.4) as the following two cases:

If $k \in I$, the inequality (3.4) is obviously true.

If $k \in J$, from Lemma 1, (2.4) and (3.7), we have

$$f_{k+1} \leq f_k + \frac{\delta}{2}(1 - \frac{\delta\tau}{\upsilon})(-g_k^T s_k) \leq C_k - \frac{\delta_1 \delta}{2}(1 - \frac{\delta\tau}{\upsilon}) \|g_k\| \min\{\Delta_k, \frac{\|g_k\|}{\|B_k\|}\} \tag{3.8}$$

We combine (3.5) and (3.8), we obtain $f_{k+1} \leq C_k - \eta \|g_k\| \min\{\Delta_k, \|g_k\| / \|B_k\|\}$.

So, $$C_{k+1} = \frac{\eta_k Q_k C_k + C_k - \eta \|g_k\| \min\{\Delta_k, \|g_k\| / \|B_k\|\}}{Q_{k+1}} \leq C_k - \frac{\eta \|g_k\| \min\{\Delta_k, \|g_k\| / \|B_k\|\}}{Q_{k+1}} \tag{3.9}$$

From above inequality, the Lemma is true.

Lemma 4. (See Lemma 3.2 in [12]) Suppose that A2 holds, there exists a $\bar{c} > 0$ we have

$$\|s_k\| \leq \bar{c} \|g_k\| \tag{3.10}$$

Lemma 5. (See Lemma 3.6 in [13]) Suppose that A1 holds, and there is a positive number $\omega > 0$ such that $\|g_k\| \geq \omega$ for all k, then there exists a $\bar{\Delta} > 0$, such that for all k, we have $\Delta_k \geq \bar{\Delta}$.

Theorem 6. Suppose that A1-A4 hold and $\{x_k\}$ satisfies $\sum_{k=0}^{\infty} \frac{1}{M_k} = \infty$ $(M_k = 1 + \max_{1 \leq i \leq k} \|B_i\|)$. Let the sequence $\{x_k\}$ generated by Algorithm 2.1, then we have

$$\liminf_{k \to \infty} \|g_k\| = 0. \tag{3.11}$$

Proof. The proof is similar to Theorem 3.7

CONCLUSIONS

In this paper, we present a new non-monotone conic trust region method with fixed step. Under some mild conditions, we proved the global convergence result of the proposed method.

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