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Observations on the Hyperbola $y^2=120x^2+1$

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Abstract: The binary quadratic equation representing the hyperbola $y^2 = 120x^2 + 1$ is analyzed for its distinct integer solutions. A few properties among the solutions are presented Employing the integral solutions of the quadratic equation under consideration, a special Pythagorean triangle is obtained.

Keywords: Binary quadratic, Hyperbola, Integral solutions, Pell equation 2010 Mathematics subject classification: 11D09.

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non- square positive integer, has been studied by various mathematicians for its non- trivial integral solutions when D takes different integral values [1-4]. In [5] infinitely many Pythagorean triangle in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation of $y^2 = 3x^2 + 1$. In [6], a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$.

In [7], different patterns of infinitely many Pythagorean triangle are obtained by employing the non-trivial solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [8-16]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 120x^2 + 1$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under the consideration, a special Pythagorean triangle is obtained.

METHOD OF ANALYSIS:

The binary quadratic equation representing hyperbola under consideration is

$$y^2 = 120x^2 + 1 \qquad (1)$$

whose general solution (x_n, y_n) is given by

$$x_n = \frac{g_n}{2\sqrt{120}}, \quad y_n = \frac{f_n}{2},$$

where

$$f_n = (11+\sqrt{120})^{n+1} + (11-\sqrt{120})^{n+1}$$
$$g_n = (11+\sqrt{120})^{n+1} - (11-\sqrt{120})^{n+1}, \text{ n=0, 1 ,2, 3,.........}$$

The recurrence relations satisfied by x and y are given by,

$$y_{n+2} - 22y_{n+1} + y_n = 0, \; y_0 = 11, y_1 = 241$$
$$x_{n+2} - 22x_{n+1} + x_n = 0, \; x_0 = 1, x_1 = 22$$

A few numerical examples are given below

N	x_n	y_n
0	1	11
1	22	241
2	483	5291
3	10604	116161
4	232805	2550501
5	5111106	55989361
6	112211527	1229215691

From the above table we observe some interesting properties

1. $x_{2n-1} \equiv 0 (\text{mod } 2)$
2. $x_{n+1} = 11x_n + y_n$
3. $x_{n+2} = 241x_n + 22y_n$
4. $x_{n+2} = 22x_{n+1} - x_n$
5. $11x_{n+2} = 241x_{n+1} + y_n$
6. $y_{n+1} = 11y_n + 120x_n$
7. $y_{n+2} = 241y_n + 2640x_n$
8. $11y_{n+2} = 120x_n + 241y_{n+1}$
9. $y_{n+2} = 22y_{n+1} - y_n$
10. $t_{3,\frac{y_n - 1}{2}} = 15x_n^2$, $t_{3,n}$ is the triangular number.

REMARKABLE OBSERVATIONS:

I: Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table I below:

Table I:

X	Y	PARABOLA
x_n	$2x_{2n+2} - 22x_{2n+1}$	$Y = 480X^2 + 2$
x_n	$2x_{2n+3} - 241x_{2n+1}$	$Y = 5280X^2 + 22$
x_n	$y_{2n+2} - 120x_{2n+1}$	$Y = 2640X^2 + 11$
x_n	$y_{2n+3} - 2640x_{2n+1}$	$Y = 57840X^2 + 241$

II: Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table II below:

Table II:

X	Y	HYPERBOLA
$x_{n+1} - y_n$	y_n	$484Y^2 = 480X^2 + 484$
$x_{n+2} - 22y_n$	y_n	$232324Y^2 = 480X^2 + 232324$
x_n	$y_{n+1} - 120x_n$	$Y^2 = 14520X^2 + 121$
x_n	$y_{n+2} - 2640x_n$	$Y^2 = 6969720X^2 + 58081$
x_n	$x_{n+2} - 241x_n$	$Y^2 = 58080X^2 + 484$
x_n	$x_{n+1} - 11x_n$	$Y^2 = 120X^2 + 1$
$y_{n+1} - 11y_n$	y_n	$480Y^2 = 4X^2 + 480$
$y_{n+2} - 241y_n$	y_n	$58080Y^2 = X^2 + 58080$

III: Let m, n be two non-zero distinct positive integers such that, $m = x_n + y_n, n = x_n$. Note that $m > n > 0$. Treat m, n as the generators of the pythagorean triangle $T(\alpha,\beta,\gamma)$. where $\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$.

Let A,P represent the area and perimeter of $T(\alpha,\beta,\gamma)$. Then the following relations are observed.

(a) $\alpha - 60\beta + 59\gamma = -1$

(b) $\alpha + \beta - \gamma = \frac{4A}{P} = 2x_n y_n$

(c) $\frac{4A}{p} - 61\beta + 60\gamma = -1$

CONCLUSION

In this paper, we have presented infinitely many integer solutions to the hyperbola $y^2 = 120x^2 + 1$ .To conclude, one may search for other choices of hyperbola for patterns of solutions and their corresponding properties.

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