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Alarge-Update Primal-Dual Interior-Point Algorithm for Linear Optimization Problem Based on a Trigonometric Kernel Function

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Abstract: In this paper, an interior point algorithm for linear optimization problem based on a kernel function which has trigonometric function in its barrier term is proposed. By means of some simple analysis tools, we show that our algorithm in large neighborhood of the central path has the best known iteration complexity bound.
Keywords: Trigonometric Kernel Function, linear optimization, algorithm.

INTRODUCTION

We consider the primal problem of linear optimization (LO) problems in the standard form as below

$$\min\{c^T x, Ax = b, x \geq 0\} \quad \text{(P)}$$

and its dual problem is given by:

$$\max\{b^T y : A^T y + s = c, s \geq 0\} \quad \text{(D)}$$

where, $A \in \Re^{m\times n}$, $x, c \in \Re^n$ and $y \in \Re^m, s \in \Re^n$. Moreover, we assume that $rank(A) = m$.

An efficient interior point algorithm for solving LO problem was first suggested by Karmarkar [7] in 1984. Then Kojima *et al.* [8] and Megiddo in [10] proposed primal dual interior point algorithm for LO problem for the first time. In recent years, several algorithms for solving convex optimization problems have been constructed [13-15]. Recently, Peng *et al*. Proposed primal-dual interior point algorithm for LO problem based on the so-called Self-Regular (SR) kernel functions [15]. They extended their algorithm to other convex optimization problems, so the currently best known iteration complexity bounds for large and small-update methods i.e., $o(\sqrt{n}\log n \log\frac{n}{\varepsilon})$ and $o(\sqrt{n}\log\frac{n}{\varepsilon})$ respectively, has been derived. Recently, several kernel function have been introduced. For instance, we can refer to works proposed in [1-5]. Several kernel function with trigonometric barrier term have been introduced [5,6,16,17,19-25].

In this paper, we present a large-update primal dual interior point algorithm for LO problem based on the following trigonometric kernel function:

$$\varphi(t) := \frac{t^2-1}{2} - \int_1^t \tan(h(x))e^{3p(\tan(h(x))-1)}dx, \quad h(x) = \frac{\pi}{2+2x}, \text{ where } p \geq 1. \qquad (1)$$

This function was first studied by Fathi *et al*. [25] for second order cone optimization problems. Based on this kernel function, we study the primal-dual interior point method (IPM) for solving LO problems and derived the iteration complexity bound for large-update method as the best known iteration bounds.

Central path for LO

In this section, some concepts of primal dual and the central path for LO problems recalled briefly. Finally, we present the generic interior point algorithm for LO. Throughout the paper, we assume that the both problems (P) and (D) satisfy the interior point condition (IPC), that is, there exists (x^0, y^0, s^0) such that $Ax^0 = b, x^0 > 0, A^T y^0 + s^0 = c, s^0 > 0$.

It is well known that the Karush-Kuhn-Tucker (KKT) conditions for both problems (P) and (D) are

$$\begin{aligned} &Ax = b, x \geq 0, \\ &A^T y + s = c, s \geq 0 \qquad (2) \\ &xs = 0. \end{aligned}$$

The key idea behind primal-dual IPMs for solving LO problems is to replace the third equation in (2) by the parameterized nonlinear equation $xs = \mu e$, where $\mu > 0$. Therefore, the system (2) can be rewritten as

$$\begin{aligned} &Ax = b, x \geq 0, \\ &A^T y + s = c, s \geq 0 \quad (3) \\ &xs = \mu e. \end{aligned}$$

Note that, this system has a unique solution as $(x(\mu), y(\mu), s(\mu))$, where $x(\mu)$ is called the μ centeral of (P) and $(y(\mu), s(\mu))$ the μ center of (D). The set of all μ-centers (with μ running through of all positive real number), gives a homotopy path [11], which is the so-called central path for both problems (P) and (D) [11, 12].
Applying Newton's method to the system (3), we obtain the following Newton-system:

$$\begin{aligned} &A\Delta x = 0 \\ &A^T \Delta y + \Delta s = 0 \qquad (4) \\ &x\Delta s + s\Delta x = \mu e - xs. \end{aligned}$$

Note that this system has a unique solution [12]. Now we can derive the new point as:
$x_+ = x + \alpha\Delta x,\ \ y_+ = y + \alpha\Delta y,\ \ s_+ = s + \alpha\Delta s,$ where $\alpha \in (0,1]$ is the so called step size. Let us define the scaled vector ϑ as $\vartheta = \sqrt{\frac{xs}{\mu}}$ and the scaled search directions $d_x = \frac{\vartheta\Delta x}{x}$, $d_s = \frac{\vartheta\Delta s}{s}$. Now the system (4) can be rewritten as below:

$$\begin{aligned} &\bar{A} d_x = 0 \\ &\bar{A}^T \Delta y + d_s = 0 \qquad (5) \\ &d_x + d_s = \vartheta^{-1} - \vartheta, \end{aligned}$$

Where, $\bar{A} = \frac{1}{\mu} AV^{-1}X = \frac{1}{\mu} AV^{-1}S$, and $V = diag(\vartheta),\ X = diag(x),\ S = diag(s)$.

A crucial observation is that the right hand side $\vartheta^{-1} - \vartheta$ in the last equation of the system(5) equals to the minus gradient of the following proximity function $\psi_c(\vartheta) := \sum_{i=1}^{n} \varphi_c(\vartheta_i)$, where, $\varphi_c(t)$ is so called the kernel function.

Peng *et al.* in [10] replaced the proximity function ψ_c with the proximity function $\psi(\vartheta) = \sum_{i=1}^{n} \varphi(\vartheta_i)$, where thekernel function satisfies the following conditions:

- $\varphi(1) = \varphi'(1) = 0$
- The function $\varphi(t)$ is strictly convex function for all $t > 0$.
- $\lim_{t \to 0^+} \varphi(t) = \lim_{t \to +\infty} \varphi(t) = +\infty$

In this paper, we use the kernel function (1) and the new search direction is computed by solving the following system:

$$\bar{A}d_x = 0$$

$$\bar{A}^T \Delta y + d_s = 0 \qquad (6)$$

$$d_x + d_s = -\nabla \psi(\vartheta),$$

Where, $\nabla \psi(\vartheta)$ denotes the gradient proximity function, which induced by the new kernel function $\varphi(t)$ introduced by (1). Therefore, we have:

$$d_x = d_s = 0 \Leftrightarrow \nabla \psi(\vartheta) = 0 \Leftrightarrow \psi(\vartheta) = 0 \Leftrightarrow \vartheta = e.$$

The generic primal-dual algorithm for LO problem is as follows:

Algorithm 1 generic primal dual IPM for LO

Input
a proximity function $\psi(\vartheta)$
a threshold parameter $\tau > 0$
an accuracy parameter $\varepsilon > 0$
a barrier update parameter θ, $0 < \theta < 1$.
begin
$x := e;\ s := e;\ \mu := 1;\ \vartheta := e$
while $n\mu > \varepsilon$ do
begin
$\mu := (1-\theta)\mu;$
while $\psi(\vartheta) > \tau$ do
begin
$x := x + \Delta x,\ y := y + \Delta y,\ s := s + \Delta s \quad \vartheta := \sqrt{\frac{xs}{\mu}}$
end
end
end

Some properties of the kernel function

In this section some properties of the kernel function which is defined by (1) are proposed. Note that for the function defined by (1), we have:

$$\varphi'(t) = t - \tan(h(t))e^{3p(\tan(h(t))-1)}$$

$$\varphi''(t) = 1 + \frac{2\pi}{(2+4t)^2}(1+3p\tan(h(t)))r(t)e^{3p(\tan(h(t))-1)}$$

$$\varphi'''(t) = -\frac{4\pi}{(2+4t)^3}r(t)e^{3p(\tan(h(t))-1)}K(t)$$

where

$$r(t) = 1 + \tan^2(h(t))$$

$$K(t) = (2 + \frac{2\pi}{2+2t}\tan(h(t)))(1+3p\tan(h(t))) + \frac{3p\pi}{2+2t}r(t)(2+3p\tanh(t)))$$

Lemma 1: [Lemmas 3.2 and 3.3 in [25]]For the function $\varphi(t)$ defined by (1), we have:

1) $\varphi''(t) \ge 1$
2) $t\varphi''(t) + \varphi'(t) > 0$
3) $t\varphi''(t) - \varphi'(t) > 0$
4) $\varphi'''(t) < 0$

where, $t > 0$.

From Lemma 2.4 in [1] and the second part of the Lemma 1, we conclude that the kernel function defined by (1) has the exponential convexity property, which plays an important role in the complexity analysis of the algorithm. In next lemma, we present some other properties of this kernel function.

Lemma 2: [Lemma 3.4 in [17]]Let the kernel function $\varphi(t)$ be defined as in (1). Then, we have:

1) $\frac{1}{2}(t-1)^2 \leq \varphi(t) \leq \frac{1}{2}\varphi'(t)^2 \quad t > 0$

2) $\psi(\vartheta) \leq 2\delta(\vartheta)^2$

3) $\|\vartheta\| \leq \sqrt{n} + 2\delta(\vartheta),$

where, the norm-based proximity measure $\delta(\vartheta)$ is defined as follows:

$$\delta(\vartheta) := \frac{1}{2}\|\nabla\psi(\vartheta)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\varphi(\vartheta_i))^2}, \qquad \vartheta \in \Re^n_{++}.$$

In the next lemma, we studythe growth behavior of the proximity function after a μ - update.

Lemma 3: (Lemma 4.2 in [17]) Lemma Suppose that $0 < \theta < 1$ and $\vartheta_+ := \frac{\vartheta}{\sqrt{1-\theta}}$. Then, we have:

$$\psi(\vartheta_+) \leq \psi(\vartheta) + \frac{\theta}{2(1-\theta)}(2\psi(\vartheta) + 2\sqrt{2n\psi(\vartheta)} + n).$$

COMPLEXITY RESULTS

In this section, the worst iteration complexity bounds case for large-update primal-dual IPM based on the proposed kernel function are computed.

First, from the exponential convexity property of the function ψ, we have:

$$\psi(\vartheta_+) \leq \frac{1}{2}(\psi(\vartheta + \alpha d_x) + \psi(\vartheta + \alpha d_s)).$$

Let us define the functions:

$$f(\alpha) := \psi(\vartheta_+) - \psi(\vartheta)$$

$$f_1(\alpha) := \frac{1}{2}(\psi(\vartheta + \alpha d_x) + \psi(\vartheta + \alpha d_s)) - \psi(\vartheta)$$

Lemma 4: (Lemma 4.3 in [1]) Let $\rho[0,\infty) \to (0,1]$ be the inverse of the function $-\frac{1}{2}\varphi'(t)$ in the interval (0, 1]. Then, the largest possible value for α in order to satisfy (17) is given by

$$\bar{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)). \qquad (7)$$

Lemma 5: (Lemma 4.4 in [1]) Let α be defined as in (7). Then, it yields the following inequality:

$$\bar{\bar{\alpha}} \geq \frac{1}{\varphi''(\rho(2\delta))}. \qquad (8)$$

Lemma 6: (Lemma 3.5 in [4]) Suppose that the step size α is such that $\alpha \leq \bar{\alpha}$, thus we have:

$$f(\alpha) \leq -\alpha\delta^2$$

In the next lemma, we present the decrease in the proximity function during an inner iteration by considering the default value for the step size, i.e., $\alpha = \bar{\bar{\alpha}}$.

Lemma 7: [Lemma in [25]] Let $\psi(\vartheta) \geq 1$, and ρ and $\bar{\bar{\alpha}}$ be defined as in Lemma 4 and (8), respectively. Then, we have:

$$f(\alpha) \leq -\frac{\delta^2}{\varphi''(\rho(2\delta))} \leq -\Theta\left(\frac{\delta}{p(1+\frac{1}{p}\log(4\delta+1))^2}\right).$$

Now, we can compute the iteration bound for the total number in an outer iteration of the Algorithm 1. Let us denote the value of $\psi(\vartheta)$ after μ-update by ψ_0, and the subsequent values by ψ_j, for $j = 1, 2, ..., L-1$, where L is the total number of inner iterations in an outer iteration. Therefore, we have:

Lemma 8:Let μ be updated by $\mu := (1-\theta)\mu$ and $\tau \geq 1$. Then, the number of inner iterations that are required to return the iterations back to the situation where $\psi(\vartheta) \leq \tau$ is bounded above by:

$$L \leq 1 + \frac{2p(1+\frac{1}{p}\log\psi_0)^2}{\kappa}\psi_0^{\frac{1}{2}}.$$

Due to (Lemma 3), for the large update interior point methods, we have $\psi_0 = O(n)$. Therefore, an upper bound for the total number of inner iterations in an outer iteration is derived as below:

$$L \leq 1 + \frac{2p(1+\frac{1}{p}\log\psi_0)^2}{\kappa}\psi_0^{\frac{1}{2}} = 0(p\sqrt{n}(1+\frac{1}{p}\log n)^2).$$

From Lemma I.36 in [18], the total number of outer iterations for achieving $n\mu < \varepsilon$ are bounded above by $O(\frac{1}{\theta}\log\frac{n}{\varepsilon})$.

On the other hand, the total number of iterations for the Algorithm 1is derived by multiplying the total number of inner and outer iterations. Note that in largeupdate method, we select $\theta = \Theta(1)$. Therefore, the total number of iterations to get a solution, i.e. a solution that satisfies $x^T s = n\mu < \varepsilon$, is given by:

$$0(p\sqrt{n}(1+\frac{1}{p}\log n)^2\log\frac{n}{\varepsilon}).$$

By putting $p = O(\log n)$, the iteration complexity bound for large-update methods is given by:

$0(\sqrt{n}\log n\log\frac{n}{\varepsilon})$, which matches with the currently best-known iteration bound for large update methods**.**

CONCLUDING REMARKS

In this paper, a large-update primal dual IPMs for solving LO problems based on the trigonometric kernel function has been proposed. By using some mild and easy to check conditions, a simple analysis for the primal dual IPMs based on the proximity function induced by the new kernel function is provided. Finally, we prove that the primal-dual IPM based on this kernel function has $0(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ iteration complexity bounds, which matches with the current best-known iteration bound for large update methods.

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