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# A Simple Pseudo Inversion Method for Solution to PDE with Financial Application

**Okechukwu U. Solomon**
Department of Physical Science, Rhema University Aba, Abia State, Nigeria.

***Corresponding Author:**
Okechukwu U. Solomon
Email: okesolo2000@yahoo.co.uk

**Abstract:** A new effective method for pricing American option under the Black-Scholes model using pseudo inversion matrix is proposed and justified. The bases for a pseudo inversion matrix to price an American Option under the Black-Scholes model depend on the complete understanding of the discretization process and stability principles. For the past years, a lot of researchers have applied pseudo inversion matrix in solving system equations, like reconfigurable control system (RCS), Hermitian differential-algebraic system etc. This paper presents a simple pseudo inversion method for American option valuation under Black-Scholes model, through a drifted financial derivative system, discretized from Black-Scholes financial PDE. An illustrative example is given in concrete setting.
**Keywords:** Financial PDE; Stochastic algorithm; Drifted financial derivative system; Option pricing; Central finite difference discretization; Pseudo inversion matrix.

## INTRODUCTION

For the past couple of decades, researchers have been investigating extensively the problem of pricing American options both in numerical methods and analytical approximations. These two approaches both encounter the same intersection of challenge which is known as a free boundary value problem arising from the early exercise feature of American options. The difficulty associated with the valuation of American options stems from the fact that the optimal exercise boundary must be determined as a part of the solution. Unfortunately, the early exercise boundary cannot be solved in a close form, and, consequently, nor can the option's value. Up to now, reviewing all relevant literatures, most efforts have been exerted mainly on locating the free boundary that sets the domain for the Black-Scholes equation [1-3]. For pricing options, a model is needed for the behavior of the value of the underlying asset. Many such models of varying complexity have been developed. More complicated models reproduce more realistic paths for the value and match between the market price and model prices of options, but they also make pricing more challenging. Black and Scholes [4] discovered the partial differential equation which financial derivatives (the underlying assets of which are stocks and bond) have to satisfy; furthermore, they found the evaluation formula when the financial derivative is a European call option. The partial differential equation is known as the Black-Scholes equation. Scholes obtained a Nobel Prize for economics in 1997 for this contribution. In this paper, we use Black-Scholes model to ascertain the behavior of the underlying asset.

Since trading of options have grown to a tremendous scale during the last decades the need for accurate and effective numerical option pricing methods is obvious. For numerical approximations, the most popular numerical methods for pricing American options can be classified to lattice method, Monte Carlo simulation and finite difference method. Sure, besides finite difference methods, there are other popular numerical method based on discretization for solving PDEs like finite element method, boundary element method, spectral and pseudo-spectral methods and etc. Usually American options need to be priced numerically due to the early exercise possibility. One approach is to formulate a linear complementarity problem (LCP) or variational inequality with a partial (integro-) differential operator for the price and then solve it numerically after discretization. Another way is to discretize the Black-Scholes differential equation into system of ordinary differential equation and further transform into a drifted financial derivative system and then solve numerically using stochastic approximation method and Pseudo-Inverse Method (PIM). In the failure detection and failure identification areas for the past years, numerous approaches have been developed to control law reconfiguration. One of them, the Pseudo-Inverse Method (PIM), has been accepted as a key approach to reconfigurable control and it has been used quite successfully in flight simulations as reported by [5-8]. The main idea is to modify the feedback gain so that the reconfigured system approximates the nominal system in some sense. In this paper, Pseudo-Inverse Method is propose to price an American Option under the Black-Scholes model because of its attractive nature and simplicity in computation and implementation.

The outline of the paper is the following: In section 2 we review modeling of Black-Scholes, the partial differential equation which financial derivative have to satisfy and we discretize the generic PDE into LCP and drift financial derivative system for American option valuation. In section, 3, controllability and stability of a financial derivative system for pseudo-inverse method is presented. Numerical experiments are presented in section 4 and conclusions are given in section 5.

**OPTION PRICING MODEL**

Here, we consider the Black-Scholes Model [4] and Merton [9] and the partial differential equation which financial derivative (stock) have to satisfy. The Black-Scholes Model assumes a market consisting of a single risky asset (S) and a risky-free bank account (r). This market is given by the equations:

$$dS = \mu S dt + \sigma S dZ, \qquad (1)$$

called the geometric Brownian-Motion and

$$dB = rBdt, \qquad (2)$$

called the non-stochastic, where $Z$ is Brownian motion, $B$ is the bond value and the interpretation of the parameters is as follows:

$\mu$ is the expected rate of return in the risk asset (drift),

$\sigma > 0$ is the volatility of the risky asset,

$r \geq 0$ is the bank's rate of interest.

In this form $\mu$ is the mean return of S, and $\sigma$ is a variance. The quantity $dZ$ is a random variable having a normal distribution with mean 0 and variance $dt$.

$$dZ \propto N(0, (\sqrt{dt})^2) \, .$$

For each interval $dt$, $dZ$ is a sample drawn from the distribution $N(0, (\sqrt{dt})^2)$, and multiplied by $\sigma$ to produce the term $\sigma dZ$. The value of the parameters $\mu$ and $\sigma$ may be estimated from historical data.

Under the usual assumptions, Black and Scholes [4] and Merton [9] have shown that the worth $V(t, S)$ of any contingent claim written on a stock, whether it is American or European, satisfies the famous Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\,\frac{\partial V}{\partial S} - rV = 0, \qquad (3)$$

where volatility $\sigma$, the risk-free rate $r$, and dividend yield $q$ are all assumed to be constants. The value of any particular contingent claim is determined by the terminal and boundary conditions. For an American option, notice that the PDE only holds in the not-yet-exercised region. At the place where the option should be exercised immediately, the equality sign in (3) would turn into an inequality one. That means the option value $V(t, S)$ at each time follows either

$V(t, S) = \Lambda(t, S)$ for the early exercised region or (3) for the not-yet-exercised region, where

$\Lambda(t, S)$ is the payoff of an American option at time t.

The generic form of (3) is derived by the change of variable $\tau = T - t$ to

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \;\frac{\partial^2 V}{\partial S^2} - (r - q)\,S\,\frac{\partial V}{\partial S} + rV = 0\,, \qquad (4)$$

where $V(\tau, .) \equiv V(T - \tau, .)$, $\sigma\,(\tau, .) \equiv \sigma\,(T - \tau, .)$, $0 \leq \tau \leq T$

$S_{min} < S < S_{max}$, subject to the initial condition:

$$V(0, S) = \Lambda(S). \qquad (5)$$

For the computations, the unbounded domain is truncated to

$$(t, S) \in (S, 0) \times (0, T]\,, \qquad (6)$$

with sufficiently large $S \equiv S_{max}$.

The worth V of an American option under Black-Scholes model satisfies an LCP

$$\begin{cases} LV \geq 0 \\ V \geq \Lambda \\ (LV)\,(V-\Lambda) = 0\,, \end{cases} \qquad (7)$$

we impose the boundary conditions

$$\begin{cases} V(t,0) = 0 \\ \qquad\qquad\qquad\qquad S \in (0, S_{max}) \\ V(t,S) = \Lambda(t,S), \end{cases} \qquad (8)$$

Beyond the boundary $S = S_{max}$, the worth $V$ is approximated to be the same as the payoff $\Lambda$, that is $V\,(t,S) = \Lambda(S)$ for $S \geq S_{max}$.

**DISCRETIZING THE FINANCIAL PDE**

American options can be exercised at any time before expiry. Formally, the value of an American put option with a strike price k is

$$V(0,k) = \sup(0 \leq \tau^* \leq T: E(e^{-\tau\tau^*}(k - S_{\tau*})^+). \qquad (9)$$

The optimal exercise time $\tau^*$ is the value that maximizes the expected payoff - any scheme to price an American options must calculate this $(\tau^*)$.

For American options with payoff $\Lambda(S)$, the equivalent of equation (4) is

$$\begin{bmatrix} \frac{\partial V}{\partial \tau} - \frac{1}{2}\ \sigma^2 S^2\ \frac{\partial^2 V}{\partial S^2} - (r-q)S\ \frac{\partial V}{\partial S} + rV \geq 0 \\ V(T,S) \geq \Lambda(S) \end{bmatrix}$$

$$\left[\frac{\partial V}{\partial \tau} - \frac{1}{2}\ \sigma^2 S^2\ \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV\right][V(T,S) - \Lambda(S)] = 0. \qquad (10)$$

Consider a uniform spatial mesh on the interval $[S_{min},\ S_{max}]$:

$$S_j = S_{min} + j\delta S,\ j = 0,1,\dots,n+1, \qquad (11)$$

where

$$\delta S = \frac{S_{max} - S_{min}}{n+1}, \text{ and} \qquad (12)$$

$$S_{max} = S_0 \exp\left[\left(r - q - \frac{\sigma^2}{2}\right)T + 6\sigma\sqrt{T}\right]. \qquad (13)$$

The truncated domain (6) has the lower bound $S_{min} = 0$ and upper bound $S_{max}$ as in (13).

Replacing all derivatives with respect to S by their central finite-difference approximations, we obtain the following approximation to the Black-Scholes PDE (4)

$$\frac{\partial V(\tau,S)}{\partial \tau} = \frac{1}{2}\sigma^2\, S^2 \frac{V(\tau, S_j+\delta S) - 2V(\tau,S_j) + V\,(\tau, S_j-\delta S)}{\delta S^2} + (r-q)S\frac{V(\tau, S_j+\delta S) - V\,(\tau, S_j-\delta S)}{2\delta S}$$
$$-rV\,(\tau,S) + O(\delta S^2). \qquad (14)$$

Let $V_j\,(\tau)$ denote the semi-discrete approximation to $V(\tau, S_j)$. Applying (14) at each internal node $S_j$, we obtain the following system of first-order ordinary differential equations;

$$\frac{dV_j\,(\tau)}{d\tau} = \underbrace{\frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - \frac{(r-q)S_j}{\delta S}\right)}_{L_{j,j-1}} V_{j-1}(\tau) + \underbrace{\left(-\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - r\right)}_{L_{j,j}} V_j(\tau) + \underbrace{\frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 + \frac{(r-q)S_j}{\delta S}\right)}_{L_{j,j+1}} V_{j+1}(\tau)$$

$$j = 1, 2, \dots, n\,; \qquad (15)$$

with discretized form given as

$$\frac{dV_j(\tau)}{d\tau} = L_{j,j-1}V_{j-1}(\tau) + L_{j,j}V_j(\tau) + L_{j,j+1}V_{j+1}(\tau).$$

System (15) has n equations in $n+2$ unknown functions, $V_0(\tau), V_1(\tau), \dots, V_n(\tau), V_{n+1}(\tau)$. Using Dirichlet boundary conditions we have the functions $V_0(\tau)$ and $V_{n+1}(\tau)$ which respectively approximate the solution at the boundary nodes $S_0 = S_{min}$ and $S_{n+1} = S_{max}$. As a result, the system of differential equations (15) can be written as the following matrix-vector differential equation with an n-by-n tri-diagonal coefficient matrix L whose entries are defined in (15)

$$\frac{d\boldsymbol{V}(\tau)}{d\tau} = L\boldsymbol{V}(\tau) + \boldsymbol{G}(\tau), \qquad (16)$$

subject to the initial condition (5)

$$\boldsymbol{V}(0) = \boldsymbol{\Lambda} := [\Lambda(S_1), \Lambda(S_2), \dots, \Lambda(S_n)]' . \qquad (17)$$

Here we use the notation:

$$\boldsymbol{L} = \begin{pmatrix} L_{11} & L_{12} & 0 \cdots & 0 & 0 \\ L_{21} & L_{22} & L_{23} \cdots & 0 & 0 \\ 0 & L_{32} & L_{33} \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ \ddots & \vdots & \vdots \\ 0 & 0 & 0 \ \cdots & L_{n-1,n-1} & L_{n-1,n} \\ 0 & 0 & 0 \ \cdots & L_{n,n-1} & L_{n,n} \end{pmatrix}, \ \boldsymbol{V}(\tau) = \begin{pmatrix} V_1(\tau) \\ V_2(\tau) \\ \vdots \\ V_{n-1}(\tau) \\ V_n(\tau) \end{pmatrix}.$$

The vector $\boldsymbol{G}(\tau) \in R^n$ is given by

$$\left[\left(\frac{\sigma^2(S_0)\, S_0^2}{2\delta S^2} - \frac{(r-q)S_0}{2\delta S}\right) V_0(\tau), 0, \dots, 0, \left(\frac{\sigma^2(S_{n+1})S_{n+1}{}^2}{2\delta S^2} + \frac{(r-q)S_{n+1}}{2\delta S}\right) V_{n+1}(\tau)\right]' .$$

$\boldsymbol{G}(\tau)$ contains boundary values of the mesh solution.

The spatial discretization leads to**:**

**Semi-discrete LCP**

According to White (2013) from (10), (16) and (17), we have

$$\begin{cases} L^j\, V^{j+1} \geq g^j \\ V^{j+1} \geq \Lambda \\ (V^{j+1} - \Lambda)'(L^j\, V^{j+1} - g^j) = 0 \end{cases}, \qquad (18)$$

where L is n-by-n tri-diagonal coefficient matrix, g is a vector resulting from the second term in equation (16) V and Λ are vectors containing the grid point values of the worth V and the pay off Λ, respectively. This again must be solved at every time step. A crude approximation is to solve the system $L^j\, V = g^j$, then set $L^{j+1} = \max\ (V, \Lambda)$.

**Drifted financial derivative system**

According to Shibli [10], $G(\tau)$ term in (16) can be treated as an enforced input to the financial derivative system, resulted from boundary condition defined in (8). With zero boundary condition, equation (16) yields.

$$\dot{V} = Lv, \qquad (19)$$

which represents a pfaffian differential constraints [11], but not of kinematic nature, arises from the conservation on non-zero financial derivatives. The transformed financial derivative system (19) can be re-expressed as

$$Lv = d. \qquad (20)$$

System (20) represents a drifted financial derivative system with a drift term d. In such a system the derivative value V has been solved by Osu and Solomon [12] using stochastic algorithm, but here we propose a simple Pseudo-Inverse Method.

**CONTROLLABILITY AND STABILITY OF FINANCIAL DERIVATIVES**

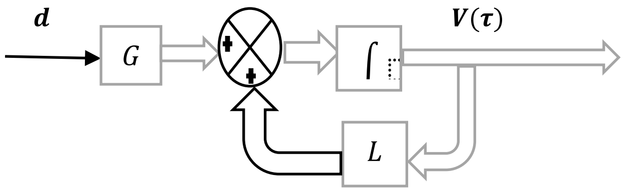


**Fig- 1: Open-loop financial controlled system**

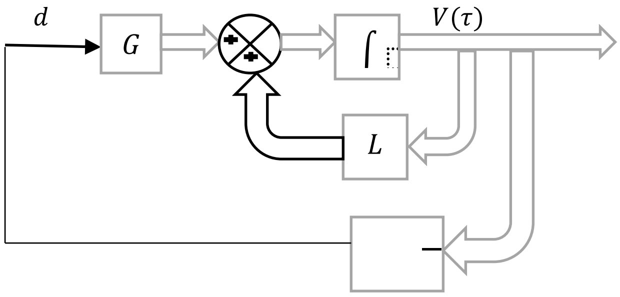


**Fig-2: Closed-loop controlled financial system with $\boldsymbol{w = -Kv}$**

In the theory of linear time-invariant dynamical control systems the most popular and the most frequently used mathematical model is given by the following differential state equation

$$\frac{dV(\tau)}{d\tau} = LV(\tau) + wG(\tau), \qquad (21)$$

where,

$V -$ *is a state vector*
$L -$ *is n x n matrix*
$G -$ *is n x* $1$
$w -$ *is a control signal*
$\tau -$ *time* .

Consider the continuous-time system shown in figure 1, the system described in Equation (21 and 16) is said to be state controllable at $\tau = \tau_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $\tau_0 \leq \tau \leq \tau_1$.

The system is said to be controllable if and only if the following n x n matrix is full rank n,

$$\widehat{M} = [G \quad LG \quad L^2G \quad \cdots \quad L^{n-1}G]. \qquad (22)$$

The matrix is called the controllability matrix.
We shall choose the control signal to be

$$w = -Kv. \qquad (23)$$

This means that the control signal is determined by an instantaneous state.

Let $\widehat{T}$ be the transformation matrix defined by;

$$\widehat{T} = \widehat{M}W, \qquad (26)$$

where $\widehat{M}$ is the controllability matrix (22) and

$$W = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a & 1 \\ a_{n-2} & a_{n-3} & \vdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}, \qquad (27)$$

where the $a_i$'s are the coefficients of the characteristic polynomial

$$|hI - L| = h^n + a_1 h^{n-1} + \cdots + a_{n-1}h + a_n. \qquad (28)$$

Let us choose a set desired eigenvalues as $h_1 = u_1, h_2 = u_2, \ldots, h_n = u_n$. Then the desired characteristic equation becomes

$$(h - u_1)(h - u_2). \cdots .(h - u_n) = h^n + \alpha_1 h^{n-1} + \cdots + \alpha_{n-1}h + \alpha_n. \quad (29)$$

The sufficient condition for the system to be completely controllable with all eigenvalues arbitrarily placed is by choosing the gain matrix

$$K = [(\alpha_n - a_n)(\alpha_{n-1} - a_{n-1}) \cdots (\alpha_2 - a_2)(\alpha_1 - a_1)]\hat{T}^{-1}. \qquad (30)$$

**THE PSEUDO-INVERSE METHOD**

Over the years pseudo-inverse of a matrix has been used by many researchers in the Reconfigurable Control system (RCS) with a considerable success. Here we applied pseudo inverse of a matrix to pricing American option as follows:

Let $L$ be a matrix as in (16 and 21), the pseudo-inverse of $L$ is defined as a matrix $L^+ \in \mathcal{M}$ ($\mathcal{M}$ is a vector space of $m \times n$ matrices) satisfying all the following criteria:

1. $LL^+L = L$ ($LL^+$ need not be the general identity matrix, but it maps all column vectors of $L$ to themselves);
2. $L^+LL^+ = L^+$ ($L^+$is a weak inverse for the multiplicative semigroup);
3. $(LL^+)' = LL^+$ ($LL^+$ is Hermitian and $(LL^+)'$ is the transpose of Hermitian); and
4. $(L^+L)' = L^+L$ ($L^+L$ is also Hermitian).

$L^+$exists for any matrix, L.

But when the latter has full rank $(n)$, $L^+$can be expressed as a simple algebraic formula.

When L has linearly independent columns (and thus matrix $L'L$ is invertible), $L^+$can be computed as:

$$L^+ = (L'L)^{-1}L', \qquad (31)$$

this particular pseudo inverse constitutes a left inverse, since, in this case, $L^+L = \mathrm{I}$.

When L has linearly independent rows (matrix $LL'$ is invertible), $L^+$can be computed as:

$$L^+ = L'(L'L)^{-1}. \qquad (32)$$

This is a right inverse, as

$$LL^+ = \mathrm{I}\ .$$

According to Shibli [10], the derivative value V can be gotten from the drifted finacial derivative system (20) by computing the pseudo inversion matrix (32) expressed in the form,

$$V = L^+d. \qquad (33)$$

However, there is one problem which might render the method useless; namely, that the solution from (33) does not necessarily make the closed- loop controlled financial system in figure 2, stable. But alternatively, the system can be stabilized if the controllability condition (22) is satisfied. Then, the pole placement design can be implemented to stabilize the system.

**NUMERICAL EXPERIMENT**

In our numerical example, we use a simple Pseudo-Inverse Method to price American put options. The parameters for the Black-Scholes model are the same as in White [13] and they are defined below:

**Table 1: Estimated parameters for the Black-Scholes model**

| Parameter | Notation | Value |
|---|---|---|
| Risk free interest rate | r | 0.2 |
| Dividend yield | q | 0.1 |
| Strike price | K | 7 |
| Volatility | $\sigma$ | 0.3 |
| Time to expiry | T | 2 |
| Spot price | $S_0$ | 10 |
| Ratio of Nodes | $\vartheta$ | 30 |

We illustrate the method in a concrete setting, by first plugging the parameters in table 1 in (11, 12 and 13), with the time nodes $3x10^3$ and space nodes $9x10^4$ satisfying the ratio of nodes $\vartheta$ as stipulated, we have $S_{max} = 142.33$, and the space discretization steps as

$$\delta S = \frac{142.33}{90001} = 0.002.$$

Thus, from (11, 12 and 15) we have;

$$S_1 = 0.002,\ L_{11} = 0.2,\ L_{12} = 0.05\ ,$$
$$S_2 = 0.004,\ L_{21} = -0.1,\ L_{22} = 0.2,\ L_{23} = 0.1,$$
$$S_3 = 0.006,\ L_{32} = -0.15,\ L_{33} = 0.2,$$

and then the financial matrix (3 by 3 tri-diagonal coefficient matrix).

$$L = \begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix}. \qquad (34)$$

By using the equation of total investment return;

$$r = d + q, \qquad (35)$$

where $r$ is the risk adjusted discount rate for V (the worth); $q$ is the dividend yield ( or convenience yield in case of commodities) and $d$ is the drift (or capital gain rate). Hence $d = 0.1$ for $q = 0.1$ and $d = 0.2$ for $q = 0.0$ (No dividend yield).

From (20), we have

$$\begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}. \qquad (36)$$

Using the financial matrix (34) in (32), we have

$$L^+ = \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix}.$$

From (33), we have

$$V = L^+ d = \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$$
$$= (0.2 \quad 0.152 \quad 0.58).$$

Applying the inversion matrix $L^+$ to (33) gives, $V_1 = (0.2 \quad 0.152 \quad 0.58)$ and $V^*(S,t) = 0.31$ for both values of the drift, and is not a fixed point and also not equal to the solution in White (2013). It is desired to check the controllability condition (22).

For $n = 3$ in (22), we have

$[G \quad LG \quad L^2G] = \widehat{M}$,

where G and L are defined as in (21) and (34) respectively.

$$L^2 = \begin{pmatrix} 0.035 & 0.02 & 0.005 \\ -0.04 & 0.02 & 0.04 \\ 0.015 & -0.06 & 0.025 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It can be easily validated that the controllability matrix

$$\widehat{M} = \begin{pmatrix} 1 & 0.25 & 0.06 \\ 1 & 0.2 & 0.02 \\ 1 & 0.05 & -0.02 \end{pmatrix}, \quad (37)$$

is of full rank 3. Since the system is controllable the pole placement design can be implemented to stabilize the system. From (28)

$$|hI - L| = \left| \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix} - \begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \right|$$

$$|hI - L| = h^3 - 0.6h^2 + 0.14h - 0.01. \quad (38)$$

With n=3 in (28), we have

$$|hI - L| = h^3 + a_1 h^2 + a_2 h + a_3. \quad (39)$$

Comparing (38) and (39), we have ; $a_1 = -0.6,\ a_2 = 0.14,\ a_3 = -0.01$.
Hence, from (27) we have

$$W = \begin{pmatrix} 0.14 & -0.6 & 1 \\ -0.6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (40)$$

Substituting (37) and (40) into the transformation matrix (26), we have

$$\hat{T} = \begin{pmatrix} 1 & 0.25 & 0.06 \\ 1 & 0.2 & 0.02 \\ 1 & 0.05 & -0.02 \end{pmatrix} \begin{pmatrix} 0.14 & -0.6 & 1 \\ -0.6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\hat{T} = \begin{pmatrix} 0.05 & -0.35 & 1 \\ 0.04 & -0.4 & 1 \\ 0.09 & -0.55 & 1 \end{pmatrix}.$$

$$|\hat{T}| = 0.004$$

$$\hat{T}^{-1} = \frac{1}{0.004} \begin{pmatrix} 0.15 & -0.2 & 0.05 \\ 0.05 & -0.04 & -0.01 \\ 0.014 & -0.004 & -0.006 \end{pmatrix}$$

$$\hat{T}^{-1} = \begin{pmatrix} 37.5 & -50 & 12.5 \\ 12.5 & -10 & -2.5 \\ 3.5 & -1 & -1.5 \end{pmatrix}. \quad (41)$$

Placing the pole $h_1 = -1,\ h_2 = -2,\ h_3 = -3$, we have the desired characteristic equations from (29) as

$$|hI - A| = \left| \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \right|,$$

where A is the pole placement diagonal matrix.

$$|hI - A| = h^3 + 6h^2 + 11h + 6\ . \quad (42)$$

From (42), we have $\alpha_1 = 6,\ \alpha_2 = 11,\ \alpha_3 = 6$.

For the system to be completely controllable with all eigenvalues arbitrarily placed we choose the gain matrix (30).

Substituting $a_i's$ and $\alpha_i's$ according to (38) and (42) and (41) into (30), we have

$$K = [(6 + 0.01) \quad (11 - 0.14) \quad (6 + 0.6)] \begin{pmatrix} 37.5 & -50 & 12.5 \\ 12.5 & -10 & -2.5 \\ 3.5 & -1 & -1.5 \end{pmatrix}$$

$$K = [384.23 \quad -415.7 \quad 38.07]\ . \quad (43)$$

Normalizing the gain matrix yields

$$K = [3.8423 \quad -4.157 \quad 0.3807]. \quad (44)$$

For a negative feedback controlled financial system as shown in Figure 2, according to Shibli (2012) it implies that to stabilize such a system, the drift parameter $d$ should increase the stock by 4.157 times (from 0.2 to 0.8), the risk free rate r should be decreased by 3.8423 times (from 0.2 to −0.77) and the volatility should also be decrease by 0.3808 times (from 0.3 to -0.11). From physical point of view, the negative sign is to balance the increase of the stock and comply with the conservation of financial money. Some systems reveal a conservation nature such as mechanical systems which comply with the principle of conservation energy.

Applying the stability condition to the pseudo inversion method (33) using the pseudo inverse $L^+$ we have

$$V = L^{+}d$$
$$= \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \end{pmatrix}$$
$$= (0.8 \quad 0.608 \quad 2.32).$$

$V_1 = (0.8 \quad 0.608 \quad 2.32)$ and $V^*(S,t) = 1.2427$ $(V^*(S,t) \approx 1.2)$. This shows that pseudo inversion method can be applied on a discretized financial PDE to price an American option and European option with a considerable success.

**CONCLUSIONS**

In this paper, the stability properties of the Pseudo- Inverse Method for pricing American options were analyzed. To guarantee stability, the controllability condition must be satisfied, before the pole placement design can be implemented to stabilize the system. For the Black-Scholes partial derivative, we employed central finite-difference approximation into first-order ordering differential equation and later transformed to a drifted financial derivative system. In numerical experiment, we formed a financial matrix and the value of the drift parameter using Table 1. With finer discretization, space nodes and time nodes, we demonstrate that the drifted financial derivative system can be efficiently and easily solved with high accuracy, by using pseudo inversion method. The pseudo inversion method proves to be simple in pricing American options, but needs the system to be stabilized for its accuracy.

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